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On Robust Dynamic Controller Design

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The general formulations of dynamic controllers are provided and two types of dynamic control schemes are developed. A design methodology has been synthesized in the time-domain. New suffiicient conditions are established for asymptotically stabilizing the dynamic controlled systems when the system has structured norm-bounded uncertainties in the continuous-time as well as in the discrete-time. Stability robustness is usually measured by the tolerance of plant matrix perturbations and the feedback control law in the time-domain. In an illustration, two dynamic control algorithms are implemented in an retail model of Industrial Dynamics to describe the design procedure.

Key Words : Dynamic Controller Design, Stability Robustness, Norm-Bounded Uncertainty, Linear Time-Invariant System

Nomenclature

λ_j : j-th eigenvalue in s-domain
z_j : j-th eigenvalue in z-domain
R : Set of real numbers
R_+ : Set of nonnegative real numbers $\{x \in R : x \geq 0\}$
R^n : Vector space of dimension n in R
$R^{n \times m}$: Matrix space with elements of n rows and m columns in R
Z_+ : Set of nonnegative integers : $\{0, 1, 2, \cdots\}$
$Re[\cdot]$: Real-part of the complex eigenvalue $[\cdot]$

Subscript

c : Continuous-time
d : Discrete-time

Superscript

e : Integrated-Error with State-Feedback (IESF) controller
s : Integrated-Error with State-Feedback and Filtering(ISFF) controller
PDR : Purchasing rate decision at retailer (units/week)
IAR : Actual Inventory at retailer(units)
UOR, UOD : Unfilled orders at retailer and distributor(units)
RRR : Requisitions(Orders) received at retailer (units/week)
RSR : Requisitions(Orders) smoothed at retailer (units/week)

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1. Introduction

In the design of control systems, it is necessary to eliminate completely the effect of offset errors caused by constant disturbances. Integral action on the dynamic controllers results in a closed-loop system in which the outputs follow step commands and reject unmeasurable arbitrary disturbances with bounded constant values. The stabilizing effect of the integral control can be counteracted by appropriate state-feedback action so that one can eventually achieve a satisfactory transient response as well as the desired zero steady-state error for arbitrary constant inputs. The pseudo-derivative feedback(PDF) control, a dynamic control with integral action, was introduced by Phelan(1977). He has suggested that the PDF controller constitutes an optimum scheme for all types of plants. The PDF control scheme of n-th order plant consists of one integrator in the feedforward loop with $(n-1)$-th order derivatives in the feedback loop. PDF demonstrates very good performance when utilized with certain low order systems but encounters serious noise effects in higher order(>3) systems. Seraji(1979) has applied PI-type controllers for multivariable systems and Krikelis(1982) has developed the PDF control scheme for 4th-order tracking problems with two PDF controllers in series included in one derivative term in the feedback loop. Maday(1987) has formalized the Integated-Error with State-Feedback(IESF) control scheme by a closed-loop pole-placement technique in the hybrid control system,which is an extension of PDF without the derivative term in the feedback loop. Recently, Aida and Kidamori(1990) has designed an optimal servo-system by a classical PI-type state-feedback control. In this paper, two types of dynamic control scheme are investigated. One is an IESF control and the other is Integated-Error with State-Feedback and Filtering (ISFF) as a new algorithm for dynamic controller. In Addition, generalized formulations about dynamic controllers are provided.

Although the dynamic control scheme has been developed to enhance system performance, stability robustness for the dynamic controlled systems has not been studied sufficiently. Robustness is usually measured by the tolerance of plant matrix perturbations in the time-domain. In the linear system with the output feedback control, sufficient robust stability conditions are derived by Sobel, Banda and Yeh(1989). Decentralized robust control for perturbed large-scale systems controlled by full-state feedback has been developed by several authors(Wang and Chang, 1989; Wu, 1989; and Ho et al., 1992). In this paper, a new sufficient condition is established for asymptotically stabilizing the perturbed systems controlled by dynamic controllers, when the system has structured norm-bounded uncertainties. Moreover, the sufficient condition for the asymptotical stability in discrete-time dynamic controlled system has been derived. This paper is divided into six parts: the formulations of the IESF and ISFF dynamic control laws are described in Sec. 2. Section 3 presents a methodology for an evaluation of ISFF controller gains using the eigen-structure. In Sec. 4, robust stability criteria for dynamic controlled systems are derived in the continuous-time domain as well as in the discrete-time domain. An algorithm and examples are shown in Sec. 5 and conclusions are provided in Sec. 6.

2. Problem Formulation

Let us consider a linear, time invariant(LTI) dynamic system as follows:

$$\dot{x}(t)=Ax(t)+Bu(t), \quad x(0)=x_0 \tag{1}$$

$$y(t)=Dx(t) \tag{2}$$

where $x(t)\in R^n$ is the state of the plant, $u(t)\in R^m$ is the control input to the plant and $y(t)\in R^l$ is the output of the plant. It is assumed that (A, B) is stabilizable and (A, D) is detectable. A, B and D are real matrices whose size is appropriate to each system, matrix B being of rank m and D of rank l. If the plant is controlled by a continuous-time controller with q-th order error dynamics, a generalized feedback and feedforward control law are described by

$$\dot{x}_r(t)=Fx_r(t)+Gx(t)+Pr(t) \tag{3}$$

$$u(t)=Rx_r(t)-Qx(t)+Cr(t) \tag{4}$$

where $x_r(t)\in R^q$ is the state vector of the dynamic controller of order q, $r(t)\in R^v$ is a reference input, F, G, P, R, Q and C are matrices of appropriate dimensions. Pure integrators of filters can be included in Eq. (3). Equation (1) augmented by Eqs. (3) and (4) yields

$$\begin{bmatrix}\dot{x}_r(t)\\ \dot{x}(t)\end{bmatrix}=\begin{bmatrix}F & G\\ BR & A-BQ\end{bmatrix}\begin{bmatrix}x_r(t)\\ x(t)\end{bmatrix}+\begin{bmatrix}P\\ BC\end{bmatrix}r(t) \tag{5}$$

Next, consider a continuous-time plant controlled by a discrete-time controller, where the sampling time is T. The plant can be discretized by

$$x(kT+T)=\Phi(T)x(kT)+\Theta(T)u(kT), \quad x(0)=x_0 \tag{6}$$

where

$$\Phi(T)=e^{AT}, \quad \Theta(T)=\int_0^T\Phi(T-\tau)Bd\tau, \quad \text{for } k=0, \cdots, \infty$$

A generalized discrete-time feedback and feedforward controller is represented by

$$x_r(kT+T)=x_r(kT)+T\{Fx_r(kT)+Gx(kT)+Pr(kT)\} \tag{7}$$

$$u(kT)=Rx_r(kT)-Qx(kT)+Cr(kT) \tag{8}$$

The closed-loop system combining the controller dynamics can be written as follows.

$$\begin{bmatrix} x_r(kT+T) \\ x(kT+T) \end{bmatrix}=\begin{bmatrix} I+TF & TG \\ \Theta R & \Phi-\Theta Q \end{bmatrix}\begin{bmatrix} x_r(kT) \\ x(kT) \end{bmatrix}+\begin{bmatrix} TP \\ \Theta C \end{bmatrix}r(kT) \tag{9}$$

where $x(kT)\in R^n$ is the state vector of the plant and $x_r(kT)\in R^q$ is the state vector of the dynamic controller. The dynamic systems including a dynamic controller as well as any conventional controllers, full-state feedback or output feedback, can be described by Eqs. (5) and (9). Two kinds of dynamic controllers are introduced in the next two sections. One is IESF control which consists of the cascaded system of PDF controllers, while the other is ISFF which consists of at least one PDF controller, full-state feedback and a first-order filter.

2.1 Integrated error with state-feedback control(IESF)

In an n-th order system, the IESF controller may be characterized by at least one forward cascaded PDF controller with the combination of an error integration and a proportional state feedback. Integral action of IESF control results in a closed-loop system in which the output follows a step command and rejects certain unmeasurable arbitary disturbances. Figure 1 shows a typical single-input single-output(SISO) IESF control block diagram of a n-th order plant. From Eq. (5), choosing F, G, R and Q suitable for continuous-time IESF control, the total system controlled by continuous-time IESF control can be described by $\dot{x}_c^e(t)=A_c^e(t)+B_c^e r(t)$, $x_c^e(0)=x_{c0}^e$,

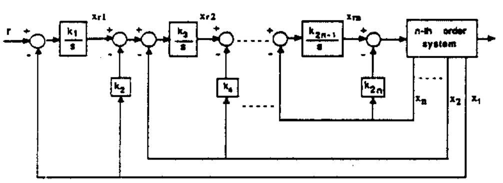

Fig. 1 IESF control of n-th order plant.

where

$$A_c^e=\begin{bmatrix} 0 & \cdots & 0 & 0 & -k_1 & 0 & 0 \\ k_3 & \cdots & 0 & 0 & -k_2k_3 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & k_{2n-1} & 0 & 0 & -k_{2n-2}k_{2n-1} & -k_{2n-1} \\ 0 & \cdots & 0 & b_1 & a_{11} & a_{1n-1} & a_{1n}-b_1k_{2n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n & a_{n1} & a_{nn-1} & a_{nn}-b_nk_{2n} \end{bmatrix},$$

$B_c^e=[k_1\ 0 \cdots 0]^T_{1\times 2n}$,

$x_c^e(t)=[x_{r1} \cdots x_{rn}\ x_1 \cdots x_n]^T$,

x_{ri} is the auxiliary error state vector of IESF, for $i=1, \cdots, n$, x_i is the state vector of the plant, for $i=1, \cdots, n$ and k_i are IESF controller gains, for $i=1, \cdots, 2n$. The controller gains $k_1, \cdots, k_{2n}$ can be chosen such that the poles of the closed-loop system A_c^s are at the desired locations in the s-plane. In moderately high-order systems, a symbolic manipulation package might be required to calculate controller gains $k_1, \cdots, k_{2n}$.

Next, consider a linear discrete-time IESF control comprised of n-th order subsystems with a sampling time T. From Eq. (9), choosing F, G, R and Q suitable for a discrete-time control, the total system becomes $x_d^e(kT+T)=A_d^e x_d^e(kT)+B_d^e r(kT)$, $x_d^e(0)=x_{d0}^e$,

where

$$A_d^e=\begin{bmatrix} 1 & 0\cdots & 0 & 0 & -k_1T & \cdots & 0 \\ k_3T & 1\cdots & 0 & 0 & -k_2k_3T & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0\cdots & k_{2n-1}T & 1 & 0 & \cdots & -k_{2n-1}T \\ 0 & 0\cdots & 0 & \theta_1 & \phi_{11} & \cdots & \phi_{1n}-\theta_1k_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0\cdots & 0 & \theta_n & \phi_{n2} & \cdots & \phi_{nn}-\theta_nk_{2n} \end{bmatrix},$$

$B_d^e=[k_1T\ 0 \cdots 0]^T_{1\times 2n}$,

$x_d^e(kT)=[x_{r1} \cdots x_{rn}\ x_1 \cdots, x_n]^T$.

In a similar manner, the controller gains $k_1, \cdots, k_{1n}$ can be determined by pole-placement such that poles are moved to desired locations.

2.2 Integrated error with state-feedback and filtering(ISFF)

The ISFF control scheme is constructed by a serial connection of at least one PDF controller, a full-state feedback and a first-order filter. If the closed-loop system is stable, ISFF control rejects

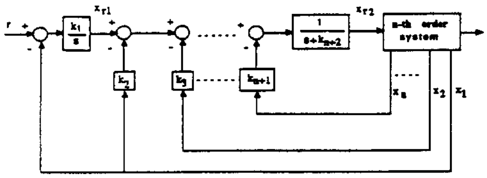

Fig. 2 ISFF control of n-th order plant.

the step disturbances(input or output) due to the integrator in the feed-forward loop. This is seen readily in the construction of total transfer function. Figure 2 shows a block diagram of a typical SISO ISFF control in the n-th order plant.

Selecting F, G, R and Q corresponding to the continuous-time ISFF control, the total system becomes $\dot{x}_c^s(t)=A_c^s x_c^s(t)+B_c^s r(t)$, $x_c^s(0)=x_{c0}^s$, where

$$A_c^s=\begin{bmatrix} 0 & 0 & -k_1 & 0 & \cdots & 0 \\ 1 & -k_{n+2} & -k_2 & -k_3 & \cdots & -k_{n+1} \\ 0 & b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & b_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$B_c^s=\begin{bmatrix} k_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n+2)\times 1}, \quad x_c^s(t)=\begin{bmatrix} x_{r1} \\ x_{r2} \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

x_{ri} is the auxillary error state vector of ISFF, for $i=1$, 2 and k_i are ISFF controller gains, for $i=1, \cdots, n+2$. Orders of the total system controlled by ISFF control are increased by two for the n-th order subsystem. The controller gains $k_1, \cdots, k_{n+2}$ can be obtained by pole-placement at the desired locations in the s-plane. In this system, one can easily determine controller gains by solving $n+2$ linear equations using the eigen-structure(Jeong, 1992).

If the subsystem is controlled by discrete-time ISFF control, the total system can be expressed by $x_d^s(kT+T)=A_d^s x_d^s(kT)$, $x_d^s(0)=x_{d0}^s$, where

$$A_d^s=\begin{bmatrix} 1 & 0 & -k_1T & 0 & \cdots & 0 \\ 1 & -k_{n+2}T & -k_2T & -k_3T & \cdots & -k_{n+1}T \\ 0 & \theta_1 & \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \theta_n & \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix},$$

$$B_d^s=\begin{bmatrix} k_1T \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_d^s(kT)=\begin{bmatrix} x_{r1} \\ x_{r2} \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

In a similar manner, the ISFF controller gains can be obtained by pole-placement at the desired locations in the z-plane. The general formulation for obtaining ISFF controller gains is derived in the next section.

3. Evaluation of ISFF Controller Gains

If the total closed-loop system matrix A_c^s in the previous section has distinct poles assigned in the stable region of the s-domain, the following form is used for obtaining ISFF controller gains :

$$(\lambda_k I-A_c^s)P_k=0, \quad \text{for } k=1, \cdots, n+2 \tag{10}$$

where λ_k is a distinct pole in the stable region in the s-domain and P_k is the eigenvector associated with λ_k.

Applying Eq. (10) for ISFF control, which makes the total order of the closed-loop system increase by 2, the following $(n+2)$-th order equation is obtained

$$\begin{bmatrix} \lambda_j & 0 & k_1 & \cdots & 0 \\ -1 & \lambda_j+k_{n+2} & k_2 & \cdots & k_{n+1} \\ 0 & -b_1 & \lambda_j-a_{11} & \cdots & -a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -b_n & -a_{n1} & \cdots & \lambda_j-a_{nn} \end{bmatrix} \begin{bmatrix} P_{1j} \\ P_{1j} \\ \vdots \\ \vdots \\ P_{n+2j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

where $j=1, \cdots, n+2$. The following submatrix is used to determine $P_{1j}, P_{2j}, \cdots, P_{n+2j}$, for $j=1, \cdots, n+2$,

$$E_{cj}P_j^c=P_{n+2j}^c \tag{12}$$

where

$$E_{cj}=\begin{bmatrix} -b_1 & \lambda_j-a_{11} & \cdots & -a_{1n-1} \\ \vdots & \vdots & \cdots & \vdots \\ -b_{n-1} & -a_{n-11} & \cdots & \lambda_j-a_{n-1n+1} \\ -b_n & -a_{n1} & & -a_{nn-1} \end{bmatrix},$$

$$P_j^c=\begin{bmatrix} P_{2j} \\ P_{3j} \\ \vdots \\ \vdots \\ P_{n+1j} \end{bmatrix}, \quad P_{n+2j}^c=\begin{bmatrix} a_{1n}P_{n+2j} \\ a_{2n}P_{n+2j} \\ \vdots \\ \vdots \\ -\lambda_j+a_{nn}P_{n+2j} \end{bmatrix}.$$

Setting one of the eigenvector elements to be equal to unity and choosing $P_{n+2j}=1$, then P_{1j}, $P_{2j}, \cdots, P_{n+2j}$ can be obtained easily by solving Eq. (12) if E_{cj} is not singular after an eigenvalue is placed at the desired location. From the first and second row of Eq. (11), one obtains

$$[\alpha_j \ P_{3j} \ P_{4j} \cdots P_{n+2j} \ P_{2j}]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n+2} \end{bmatrix}=[-\lambda_j P_{2j}] \quad (13)$$

where

$$\alpha_j=\frac{1}{\lambda_j}P_{3j}, \qquad \text{for } j=1, \cdots, n+2$$

When the closed-loop system matrix A_c^s has distinct poles in the stable region of the s-domain, the gains $k_1, \cdots, k_{n+2}$ of the continuous-time ISFF controller can be obtained by a simple matrix inversion of Eq. (13). If the system is controlled by a discrete-time ISFF controller, the gains $k_2, \cdots, k_{2n+2}$ of the disctete-time control can be determined in a similar manner. When distinct poles are placed at the desired locations in the z-domain, the two resulting equations for evaluating discrete-time ISFF controller gains are

$$E_{dj}P_j^d=P_{n+2j}^d \quad (14)$$

where

$$E_{dj}=\begin{bmatrix} -\theta_{11} & z_j-\phi_{11} & \cdots & -\phi_{1n-1} \\ \vdots & \vdots & \cdots & \vdots \\ -\theta_{n-11} & -\phi_{n-n} & \cdots & z_i-\phi_{n-1n-1} \\ -\theta_{n1} & -\phi_{n1} & & -\phi_{nn-1} \end{bmatrix},$$

$$P_j^d=\begin{bmatrix} P_{2j} \\ P_{3j} \\ \vdots \\ \vdots \\ P_{n+1j} \end{bmatrix}, \quad P_{n+2j}^d=\begin{bmatrix} \phi_{1n}P_{n+2j} \\ \phi_{2n}P_{n+2j} \\ \vdots \\ \vdots \\ -z_j+\phi_{nn}P_{n+2j} \end{bmatrix},$$

and

$$T\cdot[\beta_j \ P_{3j} \ P_{4j} \cdots P_{n+2j} \ P_{2j}]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n+2} \end{bmatrix}=[-(z_j-1)P_{2j}], \quad (15)$$

where

$$\beta_j=\frac{T}{(z_j-1)}P_{3j}, \qquad \text{for } j=1, \cdots, n+2,$$

z_j=distinct poles in the stable z-domain.

Selecting P_{n+2j} to be equal to 1, for $j=1, \cdots, n+2$, Eq. (14) provides $P_{2j}, \cdots, P_{n+2j}$. When the closed-loop system matrix A_d^s in the previous section has distinct poles in the stable z-domain, $k_1, \cdots, k_{n+2}$ of the discrete-time ISFF controller gains can be evaluated by Eq. (15).

4. Stability Robustness in Time-Domain

4.1 Preliminaries

The concepts and properties of matrix and vector norms are required for presenting stability conditions of dynamic controlled systems. Some definitions and lemmas are reviewed in this section.

Definition 1 (Desoer and Vidyasagar, 1975)
Let $x=[x_1, \cdots, x_n]\in R^n$ and $A=[a_{ij}]\in R^{n\times n}$; then $\|x\|\overset{\text{def}}{=}\sum_{i=1}^{n}|x_i|$ and

$\|A\|\overset{\text{def}}{=}\max_j\{\sum_{i=1}^{n}|a_{ij}|\}$ (column sums).

Lemma 1 (Bellman-Gronwall's Inequality) (Desoer and Vidyasagar, 195)
Let f, g ; $R_+ \to R$ and locally integrable ; if $f(t)=f$, $g(t)=g(\geq 0)$, and $v: R_+ \to R$ satisfy

$$v(t)\leq f+g\textstyle\int_0^t v(\tau)d\tau, \ \forall t\geq 0,$$
then $v(t)\leq f\exp[gt]$, $\forall t\geq 0$.

Lemma 2 (Bellman-Gronwall's Inequality for the discrete case) (Desoer and Vidyasagar, 1975)
Let v_k, f_k, h_k be real-valued sequences for $k=0, \cdots, \infty$ on the set of nonnegative integers Z_+ and $h_k\geq 0$, $\forall k\in Z_+$.
If

$$v_k\leq f_k+\sum_{0\leq i<k} h_i, \ v_i k=0, 1, 2, \cdots,$$

then $v_k\leq f_k+\sum_{0\leq i<k}\left[\prod_{i<j<k}(1+h_j)h_i f_i\right]$,

where $\prod_{i<j<k}(1+h_j)$ is set equal to 1 when $i=k-1$.

Remark From the above Lemma 2, one notes that

(a) if for some constant h_M, $h_i \le h_M$, $\forall_i$, then $v_k \le f_k + h_M \sum_{0\le i<k}(1+h_M)^{k-i-1} f_i$

(b) if for some constant f_M, $f_i \le f_M$, $\forall_i$, then $v_k \le f_M \prod_{0\le i<k}(1+h_i)$

Lemma 3 (Jeong and Maday, 1993) If a matrix $A \in R^{n\times n}$ is diagonalizable and whose eigenvalues are λ_i, for $i=1, \cdots, n$, there exists a constant $k \ge 1$ such that $\|\exp(At)\| \le k\exp(-\alpha t)$, for $t \ge 0$, $\alpha > 0$, where $-\alpha = \max_i\{R[\lambda_i(A)]\}$, for $i=1, \cdots, n$.

4.2 Robust stability in continuous-time control

Consider an LTI system with uncertainties controlled by a continuous-time dynamic controller. From Eq. (1), the continuous plant with uncertainties ΔA and ΔB is described by

$$\dot{x}(t) = (A+\Delta A)x(t) + (B+\Delta B)u(t), \quad x(0) = x_0 \tag{16}$$

It is assumed that ΔA and ΔB are linear time-invariant parametric additive perturbations with the following upper norm-bounds ; $\|\Delta A\| \le \gamma$, $\|\Delta B\| \le \zeta$. The dynamic control law can be expressed by $\dot{x}_r(t) = Fx_r(t) + Gx(t)$ and $u(t) = Rx_r(t) - Qx(t)$.

Combining the above system, the state-space form of closed-looop system is prescribed by

$$\dot{x}_c(t) = A_c x_c(t) + U_c(t), \quad x_c(0) = x_{co} \tag{17}$$

where

$$x_c(t) = \begin{bmatrix} x_r(t) \\ x(t) \end{bmatrix}, \quad A_c = \begin{bmatrix} F & G \\ BRA & -BQ \end{bmatrix},$$

$$U_c(t) = \begin{bmatrix} 0 \\ \Delta Ax(t) + \Delta BRx_r(t) - \Delta BQx(t) \end{bmatrix}.$$

The solution of Eq. (17) becomes

$$x_c(t) = \exp(A_c t)x_{c0} + \int_0^t \exp(A_c(t-\tau))U_c(\tau)d\tau \tag{18}$$

Suppose that A_c is stable and diagonalizable. From the Lemma 3, $\exp(A_c t)$ satisfies $\|\exp(A_c t)\| \le k_c \exp(-\alpha_c t)$, for $t \ge 0$, $\alpha_c > 0$, $k_c \ge 1$, where $-\alpha_c = \max_i\{Re[\lambda_i(A_c)]\}$, for $i=1, \cdots, n+q$. Since it is assumed that $\|\Delta A\| \le \gamma$ and $\|\Delta B\| \le \zeta$, it is noted that the norm of $U_c(t)$ becomes

$$\|U_c(t)\| \le \gamma\|x(t)\| + \zeta\|[R-Q]\|\|x_c(t)\|. \tag{19}$$

From the above analysis, one can obtain the following theorem concerning the robust stability of dynamic controlled system.

Theorem 1: Robust stability for dynamic Controlled system.

The parametrical perturbed system controlled by a dynamic controller is asymptotically stable($a.s.$) if F, G, R and Q are selected to satisfy the following conditions ;

a) A_c is $a.s.$,

b) $\alpha_c > k_c\{\gamma + \zeta\|[R-Q]\|\}$

proof see Appendix A.

Theorem 1 shows that in the continuous-time case, robust stability is guaranteed if the nominal eigenvalues of the dynamic controlled system lie to the left of a vertical line in the s-plane which is determined by the norm bounds associated with the structure of the uncertainty and the feedback control law(R and Q).

4.3 Robust stability in discrete-time control

Consider an LTI system controlled by a discrete-time dynamic controller with uncertainties. From Eq. (6), the discretized plant with uncertainties $\Delta\Phi$ and $\Delta\Theta$ is given by

$$x(kT+T) = (\Phi(T) + \Delta\Phi(T))x(kT) + (\Theta(T) + \Delta\Theta(T))u(kT), \quad x(0) = x_0 \tag{20}$$

It is assumed that $\Delta\Phi$ and $\Delta\Theta$ are linear time-invariant parametric additive perturbations with the following upper norm-bounds : $\|\Delta\Phi\| \le \xi$, $\|\Delta\Theta\| \le \psi$. The dynamic control law can be chosen by $x_r(kT+T) = x_r(kT) + T\{Fx_r(kT) + Gx(kT)\}$ and $u(kT) = Rx_r(kT) - Qx(kT)$.

Combining the above system, the state-space representation of closed-loop system becomes

$$x_d(kT+T) = A_d x_d(kT) + U_d(kT), \quad x_d(0) = x_{d0} \tag{21}$$

where

$$x_d(kT) = \begin{bmatrix} x_r(kT) \\ x(kT) \end{bmatrix}, \quad A_d = \begin{bmatrix} I+FT & GT \\ \Theta R & \Phi - \Theta Q \end{bmatrix},$$

$$U_d(kT) = \begin{bmatrix} 0 \\ \Delta\Phi x(kT) + \Delta\Theta Rx_r(kT) - \Delta\Theta Qx(kT) \end{bmatrix}.$$

The solution of Eq. (21) is expressed by

$$x_d(kT) = A_d^k x_{d0} + \sum_{h=0}^{k-1} A_d^{k-h-1} U_d(hT) \quad \text{for } k=0, 1, 2, \cdots \tag{22}$$

Based on the assumption that (Φ, Θ) is stabilizable, the controller gains F, G, R and Q can be determined such that A_d is asymptotically stable and its eigenvalues are distinct. Then there exist positive constants $m_e(m_e \geq 1)$ and $\tau_e(0<\tau_e<1)$ such that (Ogata, 1987)

$$0<\|A_d\|^k<m_e\tau_e^k \tag{23}$$

It is seen that the norm of $U_d(kT)$ becomes

$$\|U_d(kT)\| \leq \xi\|x(xT)\| + \psi\|[R-Q]\|\|x_d(kT)\| \tag{24}$$

From the above analysis, one can obtain the following theorem concerning the robust stability of discrete-time dynamic controlled system.

Theorem 2: Robust Stability for discrete-time dynamic controlled system

The parametrical perurbed system controlled by a discrete-time dynamic controller is asymptotically stable($a.s$) if F, G, R and Q are selected to satisfy the following conditions;

a) A_d is $a.s.$,

b) $0>\tau_e+m_e(\xi+\psi\|[R-Q]\|)<1$

proof see Appendix B.

Theorem 2 shows that in the discrete-time case, robust stability of the dynamic controlled system is guaranteed if the summations of the maximum mangitude of eigenvalues in the z-domain and the norms involving the uncertainty tolerance and the feedback control law are less than unity.

5. Algorithm and Illustrations

Based on the above analysis, an iterative algorithm is proposed to determine the dynamic controller to satisfy the robust stability condition.

Algorithms:

i) Choose the controller type and the parameters F, G, R and Q such that the closed-loop system of A_c(or A_d) is aymptotically stable.

ii) Calculate the constants k_c, α_c for continuous-time case or m_e, τ_e for discrete-time case.

iii) Evaluate the structured uncertainty upperbound.

iv) Check the robust stability condition of theorem 1(or theorem 2). If it is satisfied, then stop; otherwise, go to next step.

v) Move the poles of the closed-loop system A_c(or A_d) to the left in the s-domain in the continuous-time case or move to the origin of the z-domain in the discrete-time case. Then go to step(i).

Example:

Consider a linear nominal system of an simplified retail sector in the production-distribution system of industrial dynamics described as follows(Forrester, 1961): $\dot{X}_r(t)=A_rX_r(t)+B_ru(t)+C_rr(t)$

where $X_r(t)=[IAR\ UOD\ UOR\ RSR]^T$, $u(t)=PDR(t)$, $r(t)=RRR(t)$,

$$A_r=\begin{bmatrix}0 & 0.5 & -1.0 & 0\\ 0 & -0.5 & 0 & 0\\ 0 & 0 & -1.0 & 0\\ 0 & 0 & 0 & -0.5\end{bmatrix},\quad B_r=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad C_r=\begin{bmatrix}0\\0\\1\\0.5\end{bmatrix}.$$

Since the fixed feedforward loop and the smoothing process have no effect on the stability only if their poles are stable, consider the following subsystem to examine stability; $\dot{x}(t)=Ax(t)+Bu(t)$

where

$$x(t)=[IAR\ UOD]^T,\quad A=\begin{bmatrix}0 & 0.5\\ 0 & -0.5\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix}$$

The continuous-time IESF dynamic control law(F, G, R and Q) can be chosen by

$$F=\begin{bmatrix}0 & 0\\ 9.279 & 0\end{bmatrix},\quad G=\begin{bmatrix}-0.128 & 0\\ -5.427 & -4.640\end{bmatrix},$$
$$R=[0\ 1],\quad Q=[0\ 3.02].$$

The nominal closed-loop eigenvalues are given by $s=-0.95,\ -0.92,\ -0.85,\ -0.8$. Hence $\alpha_c=\max_i\{R_e[\lambda_i(A_c)]\}=0.8$. Assume that the structured uncertainties are bounded by $\|\Delta A\|\leq\gamma=0.3$, $\|\Delta B\|\leq\zeta=0.1$. From the theorem 1, the suffiicient condition of robust stability yields

$$\alpha_c=0.8>k_c\{\gamma+\zeta\|[R-Q]\|\}$$
$$=1\times\{0.3+0.17(3.02)\}=0.602$$

The robust stability inequality of the theorem 1 is satisfied and the robust IESF dynamic controller can stabilize the perturbed system.

Next, consider a continuous-time ISFF dynamic control. One can select the ISFF controller gains and the dynamic control law(F, G, R and Q) can be chosen by

$$F=\begin{bmatrix}0 & 0\\ 1 & -3,02\end{bmatrix},\ G=\begin{bmatrix}-1.189 & 0\\ -5.427 & -3.130\end{bmatrix},$$
$$R=[0\ 1],\quad Q=[0\ 0].$$

The nominal closed-loop eigenvalues are given by $s=-0.95,\ -0.92,\ -0.85,\ -0.8$. Hence $\alpha_c=\max_i\{R_e[\lambda_i(A_c)]\}=0.8$. In a similar manner, the sufficient condition of robust stability becomes

$$\alpha_c=0.8>k_c\{\gamma+\zeta\|[R-Q]\|\}$$
$$=1\times\{0.3+0.1\times(1.0)\}=0.4$$

The robust stability inequality of theorem 1 is satisfied and the robust ISFF dynamic controller can stabilize the perturbed system. Moreover, one can see that ISFF dynamic control has a larger robust stability margin than IESF control.

Likewisely, one can check the robust Stability of theorem 2 in the discrete-time case.

6. Conclusions

When the system is controlled by the dynamic control law, new sufficient conditions have been established for the asymptotic stability of a linear time-invariant system, subjected to structured parametric norm-bounded uncertainties in the continuous-time case as well as in the discrete-time case. In the continuous-time case, robust stability is guaranteed if the nominal eigenvalues lie to the left of a vertical line in the s-domain which is determined by the norms associated with the structure of the uncertainty and feedback control law. In the discrete-time case, robust stability is ensured if the summations of the maximum magnitude of the eigenvalues in the z-domain and the norms involving the uncertainty tolerance and the feedback control law are less than unity.

Appendices

A. Proof of theorem 1

Taking the norms on both sides of Eq. (18) and utilizing the norm inequality(19), one obtains

$$\|x_c(t)\|\le k_c\exp(-\alpha_c t)\|x_{co}\|$$
$$+\int_0^t k_c\exp(-\alpha_c(t-\tau))\{\gamma\|x(\tau)\|$$
$$+\zeta\|[R-Q]\|\|x_c(\tau)\|\}d\tau$$

since $\|x(t)\|\le\|x_c(t)\|$, then

$$\|x_c(t)\|\exp(\alpha_c t)\le k_c\|x_{co}\|+k_c\int_0^t\{\gamma$$
$$+\zeta\|[R-Q]\|\}\exp(\alpha_c\tau)\|x_c(\tau)\|d\tau$$

Applying Lemma 1 of Bellman-Gronwall's Inequality, the above equation reduces to $\|x_c(t)\|\exp(\alpha_c t)\le k_c\|x_{co}\|\exp(k_c\{\gamma+\zeta\|[R-Q]\|\}t)$, or equivalently, $\|x_c(t)\|\le k_c\|x_{co}\|\exp([-\alpha_c+k_c\{\gamma+\zeta\|[R-Q]\|\}]t)$, Hence, it is obvious that if $\alpha_c>k_c\{\gamma+\zeta\|[R-Q]\|\}$, then $\|k_c(t)\|\to 0$ as $t\to\infty$. $Q.E.D$

B. Proof of theorem 2

Taking the norms on both sides of Eq. (22) and using norm inequalities (23) and (24), one obtains

$$\|x_d(kT)\|\le m_e\tau_e^k\|x_{do}\|+\sum_{h=1}^{k-1}m_e\tau_e^{k-h-1}$$
$$\{\xi\|x(hT)\|+\psi\|[R-Q]\|\|x_d(hT)\|\}.$$

Since $\|x(kT)\|\le\|x_d(kT)\|$, and rewriting the above equations,

$$\|x_d(kT)\|\tau_e^{-k}\le m_e\|x_{do}\|+m_e\tau_e^{-1}\sum_{h=0}^{k-1}$$
$$\{\xi+\psi\|[R-Q]\|\|\}\|x_d(hT)\|\tau_e^{-h}.$$

Applying Lemma 2(b) of Bellman-Gronwall's Inequality for the discrete case, the above equation becomes

$$\|x_d(kT)\|\tau_e^{-k}\le m_e\|x_{do}\|\prod_{h=0}^{k-1}$$
$$\{1+m_e\tau_e^{-1}\{\xi+\psi\|[R-Q]\|)\},$$

or equivalently,

$$\|x_d(kT)\|\le m_e\|x_{do}\|\prod_{h=0}^{k-1}\{\tau_e+m_e(\xi+\psi\|[R-Q]\|)\}$$
$$=m_e\|x_{do}\|\{\tau_e+m_e(\xi+\psi\|[R-Q]\|)\}^k.$$

Hence, it is obvious that if $0<\tau_e+m_e(\xi+\psi\|[R-Q]\|)<1$, then $\|x_d(kT)\|\to 0$ as $k\to\infty$. $Q.E.D.$

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